Radon transforms on a class of cones with fixed axis direction

2005 J. Phys. A: Math. Gen. 38 8003

(http://iopscience.iop.org/0305-4470/38/37/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 8003–8015 doi:10.1088/0305-4470/38/37/006

Radon transforms on a class of cones with fixed axis direction

M K Nguyen[1], **T T Truong**[2] **and P Grangeat**[3]

[1] Laboratoire de Physique Théorique et Modélisation, CNRS UMR 8089, Université de Cergy-Pontoise, 2, avenue Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France
[2] Equipes Traitement des Images et du Signal, CNRS UMR 8051, Université de Cergy-Pontoise, Ecole Nationale Supérieure de l'Electronique et de ses Applications, 6, avenue du Ponceau, 95014 Cergy-Pontoise Cedex, France
[3] Laboratoire d'Electronique et de Technologie de l'Information (LETI), Département Micro Technologies pour la Biologie et la Santé (DTBS), Direction de la Recherche Technologique, Commisariat à l'Energie Atomique (CEA), CEA/GRENOBLE, 17, rue des Martyrs, 38054 Grenoble, France

E-mail: tuong.truong@ptm.u-cergy.fr

Received 18 April 2005, in final form 3 August 2005
Published 31 August 2005
Online at stacks.iop.org/JPhysA/38/8003

Abstract
Integral transforms which map functions on $\mathbb{R}^3$ onto their integrals on circular cones having fixed axis direction and variable opening angle are introduced and studied as generalizations of the known Radon transform. Besides their intrinsic mathematical interest, they serve as backbone support to emission imaging based on Compton scattered radiation, the way the standard Radon transform does for emission imaging based on non-scattered radiation. In this work, we establish its basic properties and prove analytically its invertibility. Formulae to express it in terms of the standard Radon transform (or vice versa) are given. We also discuss some extensions as applications.

PACS numbers: 02.30.Rz, 02.30.Uu, 02.30.Zz

1. Introduction

The Radon transform has attracted enormous interest in the last 50 years as it assumes a dominant role in medical imaging [1] as well as in many fields such as mathematics [10], astrophysics, geological prospecting and in particular non-destructive testing and evaluation. In his seminal paper of 1917 [15], Radon had defined an integral transform which maps a given function onto its integrals on hyperplanes of space. Subsequently there exist numerous generalizations of this standard Radon transform in which one replaces the hyperplanes by a variety of sub-manifolds in space. Of interest in many fields are quadratic surfaces such as spheres [2, 7], paraboloids [3], hyperboloids [6], etc. Generalized Radon transforms arise as integral measurements of physical quantities recorded on these surfaces. Their knowledge allows us to reconstruct the source responsible for the variations of these physical quantities in space.

In this paper we shall consider a Radon transform on circular cones in $\mathbb{R}^3$ with fixed axis orientation but with variable opening angle. They shall be called C_1-cones, to distinguish them from other types of circular cones in $\mathbb{R}^3$. The integral transform consists of 'projecting' a function f along one sheet of a C_1-cone by integrating this function f on it. The 'data' depend on the cone vertex coordinates and its opening angle as the surface of the cone sheet sweeps through space. Recall that in the standard Radon transform the plane sweeps through space by varying the direction of its normal unit vector and its distance to the origin. Technically this class of Radon transform occurs naturally in imaging processes involving Compton scattering of photons from three-dimensional sources and detected by counters along a fixed spatial direction, as proposed (but not fully treated) in [5]. An important property of this transform is its invertibility, which is most interesting for applications in particular for object reconstruction in imaging science as well as non-destructive control and/or evaluation. This will be used to give an alternative proof of the invertibility of the so-called compounded conical Radon transform, which was shown to serve as mathematical basis for a new imaging process based on the exploitation of Compton scattered radiation in emission imaging by gamma rays [12, 13].

This paper is organized as follows. Section 1 is devoted to definitions and notations to be used throughout the text. In section 2, we discuss standard properties of this transform which shall be called the C_1-Radon transform. Section 3 computes its kernel, which turns out to be a delta function concentrated on one sheet of a C_1-cone. Next we work out an alternative form of the transform which turns out to be a double Hankel transform for circular components, a technique well known in Radon transform topics. The following section deals with the proof of the invertibility of the C_1-Radon transform. We give some examples of C_1-Radon transforms of simple functions and work out its expression in terms of standard Radon integrals. An inverse to this formula exists also which permits us to express a standard Radon transform as a function of C_1-Radon transforms. As applications of this study, we show how one can invert the so-called compounded canonical Radon transform introduced sometimes ago in emission imaging by Compton scattered gamma rays. Conclusions and perspectives are given in the last section.

2. Definition and notations

We denote a linear integral transform T acting on a set of functions $f : \mathbb{R}^3 \to \mathbb{R}$ by $\widehat{Tf}_{(\ldots)}$, where $(\ldots)$ represents the set of its parameters. Thus the standard Radon transform will be called $\widehat{Rf}_{(p\mathbf{n})}$, where $\mathbf{n}$ is the normal to the plane on which f is integrated and p the distance from the origin to this plane. Similarly the Fourier transform of f is then $\widehat{Ff}_{(\mathbf{q})}$ where $\mathbf{q}$ is a vector in dual space.

In this paper, we consider a transformation defined by integrals of functions f on one sheet of C_1-cones defined by:

- a vertex S of Cartesian coordinates $(x_S, y_S, 0)$,
- an opening angle ω,
- an axis with unit vector $\mathbf{k} = (0, 0, 1)$, perpendicular to the xOy plane.

A running point V on one sheet of the cone has Cartesian coordinates: $(x = x_S + r \sin\omega \cos\psi, y = y_S + r \sin\omega \sin\psi, z = r \cos\omega)$, where r is the distance $SV (0 < r < \infty)$

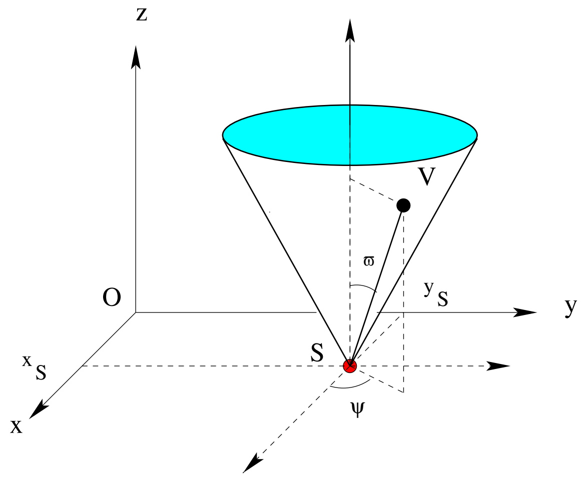

Figure 1. C_1 Radon transform with fixed axis and variable opening angle.

and ψ the polar angle $(0 < \psi < 2\pi)$ in a local cylindrical coordinate system centred at S (see figure 1).

In general f are real valued integrable functions and physical constraints require often that they have compact support. $f(V)$ is understood as its expression in Cartesian coordinates, i.e. $f(V) = f(x, y, z)$. But when cylindrical coordinates are used, this function will be denoted by $F(\rho, \theta, z) = f(\rho\cos\theta, \rho\sin\theta, z)$.

Thus the map

$$C_1 : f(V) \longmapsto \widehat{C_1 f}_{(x_S, y_S, \omega)}, \tag{1}$$

with the defining integral:

$$\widehat{C_1 f}_{(x_S, y_S, \omega)} = \int r\sin\omega\,\mathrm{d}\psi\,\mathrm{d}r f(x_S + r\sin\omega\cos\psi, y_S + r\sin\omega\sin\psi, r\cos\omega) \tag{2}$$

where $r\sin\omega\,\mathrm{d}\psi\,\mathrm{d}r$ is the integration measure on the C_1-cone, connects a function of three variables $f(x, y, z)$ to another function of three variables $\widehat{C_1 f}_{(x_S, y_S, \omega)}$.

For $0 < \omega < \pi/2$ (resp. $\pi/2 < \omega < \pi$) we deal with the forward (resp. backward) cone in the positive (resp. negative) direction of Oz. At $\omega = 0$ (resp. π) the cone becomes a positive (resp. negative) half-line parallel to Oz and the cone integral becomes a half line integral. But at $\omega = \pi/2$, this is the integral of f on the plane xOy.

As f is a real valued integrable function with compact support, $\widehat{C_1 f}_{(x_S, y_S, \omega)}$ is also of compact support with respect to (x_S, y_S) since, for given ω one can find a vertex S sufficiently far away in the xOy plane so that the cone does not intersect the support of f so that the result of the integration is zero. An estimate of the size of the support of $\widehat{C_1 f}_{(x_S, y_S, \omega)}$ can be obtained by computing the size of the circle outside which the cone of vertex S in the xOy plane is no longer tangent to the smallest sphere, for given ω containing the support of f.

3. Properties

Some simple properties are obtained directly from the above definition.

3.1. Linear transformations

The following linear transformations preserve globally the set of C_1-cones: translations, rotations in xOy and scaling by a factor $\lambda \in \mathbb{R}$ in $\mathbb{R}^3$.

Transform of translated function $f(x+\xi, y+\eta, z) = f^{(\xi,\eta)}(x,y,z)$:

$$\widehat{C_1 f}^{(\xi,\eta)}_{(x_S,y_S,\omega)} = \widehat{C_1 f}(x_S+\xi, y_S+\eta, \omega).$$

Transform of a rotated function $f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, z) = f^{(\theta)}(x,y,z)$:

$$\widehat{C_1 f}^{(\theta)}_{(x_S,y_S,\omega)} = \widehat{C_1 f}(x_S\cos\theta + y_S\sin\theta, -x_S\sin\theta + y_S\cos\theta, \omega).$$

Transform of a scaled function $f(x/\lambda, y/\lambda, z/\lambda) = f^{(\lambda)}(x,y,z)$:

$$\widehat{C_1 f}^{(\lambda)}_{(x_S,y_S,\omega)} = \lambda^2 \widehat{C_1 f}_{(x_S/\lambda, y_S/\lambda, \omega)}.$$

Proofs are simple and require only application of the definition and change of variables.

3.2. Derivatives

- The transform of a directional derivative of a function f in the direction given by an angle θ in the xOy plane,

$$\left[\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right] f(x,y,z) \quad \text{is} \quad (-)\left[\cos\theta\frac{\partial}{\partial x_S} + \sin\theta\frac{\partial}{\partial y_S}\right]\widehat{C_1 f}_{(x_S,y_S,\omega)}.$$

- The derivative with respect to the opening angle is

$$\sin\omega\frac{\partial}{\partial\omega}\left(\frac{\widehat{C_1 f}_{(x_S,y_S,\omega)}}{\sin\omega}\right) = -\int r\sin\omega\,\mathrm{d}r\,\mathrm{d}\psi\,[r(\mathbf{n}_\perp(\psi)\cdot\nabla)f(x,y,z)],$$

  where $\mathbf{n}_\perp(\psi) = (-\cos\psi\cos\omega, -\sin\psi\cos\omega, \sin\omega)$, orthogonal to the unit vector of a cone generator $\mathbf{n} = (\cos\psi\sin\omega, \sin\psi\sin\omega, \cos\omega)$. At $\omega = \pi/2$, $\mathbf{n}_\perp$ is in the direction of Oz.

4. Kernel of the transform

The defining equation (2) may be put under the standard form of a Fredholm equation of the first kind:

$$\widehat{C_1 f}_{(x_S,y_S,\omega)} = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathcal{K}(x_S,y_S,\omega|x,y,z)f(x,y,z), \tag{3}$$

with kernel $\mathcal{K}(x_S,y_S,\omega|x,y,z)$. We now show that this kernel is a delta function concentrated on one sheet of the circular cone.

By construction, this kernel is the C_1-Radon transform of a Dirac delta function concentrated at point $V = (x,y,z)$ in $\mathbb{R}^3$. It is called the PSF (point spread function) in imaging science. So we set

$$\mathcal{K}(x_S,y_S,\omega|x,y,z) = \widehat{\mathrm{PSF}}(\rho,\theta,\omega|x,y,z) = \int r\sin\omega\,\mathrm{d}\psi\,\mathrm{d}r$$
$$\times\,\delta(x_S + r\sin\omega\cos\psi - x)\delta(y_S + r\sin\omega\sin\psi - y)\delta(r\cos\omega - z). \tag{4}$$

To compute, we use the Fourier representation of the δ-function:

$$\delta(x) = \int_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{e}^{2\mathrm{i}\pi ux}, \tag{5}$$

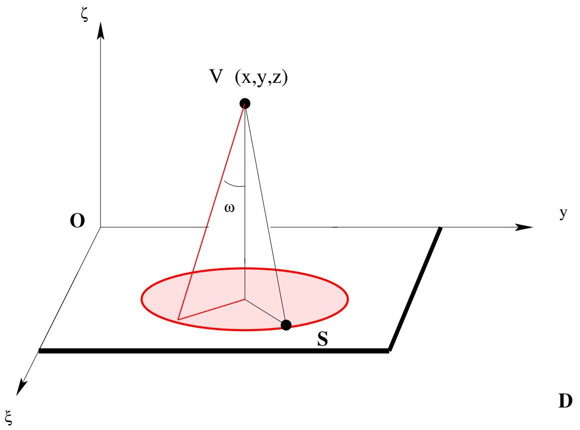

Figure 2. Representation of the kernel of the C_1 conical Radon transform.

and switch to polar coordinates for the pair of variables (u, v) (dual to (x, y)):

$$u = q\cos\beta, \qquad v = q\sin\beta, \qquad \text{and} \qquad \mathrm{d}u\,\mathrm{d}v = q\,\mathrm{d}q\,\mathrm{d}\beta,$$

as well as for the vector linking the projection of the point source on the xOy plane and the apex of the cone (x_S, y_S):

$$(x_0 - x_S) = \sigma\cos\gamma, \qquad (y_0 - y_S) = \sigma\sin\gamma.$$

Integration over $\mathrm{d}\psi$ and over $\mathrm{d}\beta$ yields the Bessel function $J_0(x)$:

$$\widehat{\mathrm{PSF}}(\rho, \theta, \omega|x, y, z) = \sin\omega \int r\,\mathrm{d}r \int q\,\mathrm{d}q\, 2\pi J_0(2\pi q r\sin\omega) 2\pi J_0(2\pi q\sigma)\delta(z - r\cos\omega). \tag{6}$$

A final integration over $\mathrm{d}r$ gives the final form of the kernel which is a delta function concentrated on one sheet of the circular cone:

$$\widehat{\mathrm{PSF}}(\rho, \theta, \omega|x, y, z) = \frac{1}{|\cos\omega|}\delta(\rho - z\tan\omega) = \delta\big(\cos\omega\sqrt{(x - x_S)^2 + (y - y_S)^2} - z\sin\omega\big), \tag{7}$$

with $\rho = \sqrt{(x - x_S)^2 + (y - y_S)^2}$. Finally the support of this delta function is, for a given point (x, y, z) and ω, a circle in the plane $x_S O y_S$ of equation:

$$\rho = /z\tan\omega.$$

(See figure 2.)

5. Analogue of the Plancherel formula

Given two functions f and g in the domain of definition of the C_1-Radon transform, one can compute the integral representing an inner product in the space of C_1-Radon transformed functions:

$$\int \widehat{C_1 f}_{(x_S, y_S, \omega)} \widehat{C_1 g}_{(x_S, y_S, \omega)}\,\mathrm{d}x_S\,\mathrm{d}y_S\,\mathrm{d}\omega = \langle\widehat{C_1 f}|\widehat{C_1 g}\rangle. \tag{8}$$

We use now the kernel form of this transform:

$$\langle\widehat{C_1 f}|\widehat{C_1 g}\rangle = \int \mathrm{d}\mathbf{r}\, f(\mathbf{r})\,\mathrm{d}\mathbf{r}'\, g(\mathbf{r}') \int \mathrm{d}x_S\,\mathrm{d}y_S\,\mathrm{d}\omega\,\delta(\cos\omega\rho - z\sin\omega)\delta(\cos\omega\rho' - z'\sin\omega), \tag{9}$$

where $\mathbf{r}$ (resp. $\mathbf{r}'$)$= (x, y, z)$ (resp. (x', y', z')) and ρ (resp. ρ') $\sqrt{(x_S - x)^2 + (y_S - y)^2}$ (resp. $\sqrt{(x_S - x')^2 + (y_S - y')^2}$). The integration on ω can be done with the variable $t = \tan\omega$ and one obtains

$$\langle \widehat{C_1 f} | \widehat{C_1 g} \rangle = \int \mathrm{d}\mathbf{r}\, f(\mathbf{r})\, \mathrm{d}\mathbf{r}'\, g(\mathbf{r}') \times \int \mathrm{d}x_S\, \mathrm{d}y_S\, \mathrm{d}\omega\, \delta\left(z\sqrt{(x_S - x')^2 + (y_S - y')^2} - z'\sqrt{(x_S - x)^2 + (y_S - y)^2}\right). \tag{10}$$

Given the points of coordinates (x, y) and (x', y') in the plane xOy, the vanishing of the argument of the delta function in equation (10) means that the point of coordinates (x_S, y_S) is on a circle, locus of points whose the distance ratio to the points (x, y) and (x', y') is given by the ratio of the ordinates z and z'. The integral is simply the length of the circle, the radius of which can be computed separately as:

$$\frac{zz'\sqrt{(x - x')^2 + (y - y')^2}}{|z^2 - z'^2|},$$

hence

$$\langle \widehat{C_1 f} | \widehat{C_1 g} \rangle = \int \mathrm{d}\mathbf{r}\, f(\mathbf{r})\, \mathrm{d}\mathbf{r}'\, g(\mathbf{r}') 2\pi \frac{\sqrt{[(x - x')^2 + (y - y')^2]zz'}}{|z^2 - z'^2|}. \tag{11}$$

Thus this inner product is reflected in the domain of the C_1-Radon transform by a metric essentially given by the radius of a circle.

6. Alternative form of the C_1 Radon transform

As in the standard Radon transform, a better insight of the C_1-Radon transform can be obtained if one applies the Fourier transform to equation (2). However, since the plane xOy plays a special role in the cone structure, we introduce the two-dimensional (or slice-) Fourier transform of f in this plane by:

$$f(x, y, z) = \int \mathrm{d}u\, \mathrm{d}v\, \mathrm{e}^{2\mathrm{i}\pi(ux+vy)} \widetilde{f}(u, v, z). \tag{12}$$

Now in order to do the circular component analysis [4], we use polar coordinates for the cone vertex ($x_S = \rho\cos\theta$, $y_S = \rho\sin\theta$) and work with $\widehat{C_1 F}_{(\rho,\theta,\omega)} = \widehat{C_1 f}_{(x_S, y_S, \omega)}$. The C_1-Radon transform of f is now

$$\widehat{C_1 F}_{(\rho,\theta,\omega)} = \int \mathrm{d}u\, \mathrm{d}v \int r\sin\omega\, \mathrm{d}\psi\, \mathrm{d}r\, \widetilde{f}(u, v, r\cos\omega) \times \exp 2\mathrm{i}\pi[u(\rho\cos\theta + r\sin\omega\cos\psi) + v(\rho\sin\theta + r\sin\omega\sin\psi)]. \tag{13}$$

But then it is more convenient to introduce polar coordinates in (u, v) space with the relations:

$$u = q\cos\beta, \qquad v = q\sin\beta, \qquad \mathrm{d}u\, \mathrm{d}v = q\, \mathrm{d}q\, \mathrm{d}\beta, \tag{14}$$

and use the function $\widetilde{F}(q, \beta, r\cos\omega) = \widetilde{f}(u, v, r\cos\omega)$.

We now assume that both $\widehat{C_1 F}_{(\rho,\theta,\omega)}$ and $\widetilde{F}(q, \beta, r\cos\omega)$ admit decompositions in circular components:

$$(\widehat{C_1 F}_{(\rho,\theta,\omega)}, \widetilde{F}(q, \beta, r\cos\omega)) = \sum_{l\in\mathbb{Z}} (\widehat{C_1 F}_{l,(\rho,\omega)} \exp \mathrm{i}l\theta, \widetilde{F}_l(q, r\cos\omega)) \exp \mathrm{i}l\beta. \tag{15}$$

Now putting these expansions in equation (2) and assuming that one can exchange the order of integrations, we perform the ψ and β integrations to obtain the Bessel functions $J_0(r)$ and $J_l(r)$. Thus an alternative form of the C_1-Radon transform relating the circular components $\widetilde{F}_l(q, r\cos\omega)$ to $\widehat{C_1 F}_l(\rho,\omega)$ appears as

$$\widehat{C_1 F}_{l,(\rho,\omega)} = \sin\omega \int_0^\infty r\,\mathrm{d}r \int_0^\infty q\,\mathrm{d}q\, \mathrm{i}^{-l} 2\pi J_l(2\pi\rho q) 2\pi J_0(2\pi q r \sin\omega) \widetilde{F}_l(q, r\cos\omega). \quad (16)$$

The circular component of the C_S Radon transformed function is thus the double Hankel transform (defined by the product of Bessel functions of order zero and of order l) of the circular component of the Fourier transformed function.

7. Inverse transform

The interesting question to ask is how to recover f if $\widehat{C_1 f}$ is given. This amounts to inverting equation (13). To perform this inversion, one uses an appropriate set of variables [5] and applies the Hankel identity for Bessel functions of order l (with $l = 0, 1, 2, \ldots$):

$$\frac{1}{k}\delta(k - k') = \int_0^\infty r\,\mathrm{d}r\, 2\pi J_l(2\pi k r) 2\pi J_l(2\pi k' r). \quad (17)$$

The first step consists of multiplying for $q' > 0$ both sides of equation (16) by

$$\int_0^\infty \rho\,\mathrm{d}\rho\, 2\pi J_l(2\pi\rho q')$$

and integrating with respect to ρ:

$$\mathrm{i}^l \int_0^\infty \rho\,\mathrm{d}\rho\, 2\pi J_l(2\pi\rho q') \widehat{C_1 F}_{l,(\rho,\omega)} = \sin\omega \int_0^\infty r\,\mathrm{d}r\, 2\pi J_0(2\pi q' r \sin\omega) \widetilde{F}_l(q', r\cos\omega). \quad (18)$$

Then the right-hand side may be viewed as the zeroth-order Hankel transform of $\widetilde{F}_l(q', r\cos\omega)$ in the following variables:

$$z = r\cos\omega \qquad \text{and} \qquad t = \tan\omega,$$

of the function

$$\widehat{G}_l(\rho, t) = \frac{\cos^2\omega}{\sin\omega} \widehat{C_1 f}_{l,(\rho,\omega)},$$

or

$$\mathrm{i}^l \int_0^\infty \rho\,\mathrm{d}\rho\, 2\pi J_l(2\pi\rho q') \widehat{G}_l(\rho, t) = \int_0^\infty z\,\mathrm{d}z\, 2\pi J_0(2\pi q' z t) \widetilde{F}_l(q', z). \quad (19)$$

Observe that the product zt remains always positive since simultaneously z and t are positive (resp. negative) for $\omega \in [0, \pi/2[$(resp.$]\pi/2, \pi])$. As z is the conjugate to $q't$ in the Hankel transform of order 0, we must break up the range of ω into two parts when applying the Hankel identity for inversion.

- $0 < \omega < \pi/2$ as $t > 0$ and for given $Q > 0$, we multiply equation (19) by

  $$\int_0^\infty t\,\mathrm{d}t\, 2\pi J_0(2\pi Q t)$$

  and integrate. The result is

  $$\int_0^\infty t\,\mathrm{d}t\, 2\pi J_0(2\pi Q t) \mathrm{i}^l \int_0^\infty \rho\,\mathrm{d}\rho\, 2\pi J_l(2\pi\rho q') \widehat{G}_l(\rho, t) = \frac{1}{q'^2} \widetilde{F}_l(q', Q/q'). \quad (20)$$

  This gives the circular component $\widetilde{F}_l(q', z)$ of $\widetilde{f}$ for $z > 0$.

- $\pi/2 < \omega < \pi$ now z is negative and varies from 0 to $-\infty$; this means that if we set $z = -z'$ as well as $t = -t'$ we have:

$$\int_0^{-\infty} z\, dz \cdots = \int_0^{\infty} z'\, dz' \cdots$$

thus we can conclude that

$$\int_0^{\infty} t'\, dt' 2\pi J_0(2\pi Q t') i^l \int_0^{\infty} \rho\, d\rho\, 2\pi J_l(2\pi\rho q') \widehat{G}_l(\rho, -t') = \frac{1}{q'^2} \widetilde{F}_l(q', -Q/q'). \quad (21)$$

In this case we recover the circular component $\widetilde{F}_l(q', z)$ for $z < 0$.

Now setting $Q/q' = z$, relabelling q' as q and taking into account the sign $\epsilon(z)$ of z, we can define the function:

$$\widehat{G}(\rho, \beta, \epsilon(z)t) = \sum_{l\in\mathbb{Z}} 2\pi i^l J_l(2\pi\rho q) \widehat{G}_l(\rho, \epsilon(z)t)\, e^{il\beta},$$

which serves to write the two-dimensional Fourier transform of the function to be reconstructed:

$$\widetilde{F}(q, \beta, z) = q^2 \int_0^{\infty} t\, dt\, 2\pi J_0(2\pi q z t) \int_0^{\infty} \rho\, d\rho\, \widehat{G}(\rho, \beta, \epsilon(z)t), \quad (22)$$

and the final form of the inversion formula, which makes use of equation (12) under the polar form:

$$f(x, y, z) = \int q\, dq\, d\beta \exp[2i\pi q[x\cos\beta + y\sin\beta]] \widetilde{F}(q, \beta, z). \quad (23)$$

The inversion of the C_1-Radon transform is then completed and it is clear that it makes use of all the 'data' from cone sheets above the xOy plane as well as below since $0 < \omega < \pi$.

Remark. Equation (18) becomes at $\omega = \pi/2$:

$$i^l \int_0^{\infty} \rho\, d\rho\, 2\pi J_l(2\pi\rho q') \widehat{C_1 F}_{l,(\rho,\pi/2)} = \int_0^{\infty} r\, dr\, 2\pi J_0(2\pi q' r) \widetilde{F}_l(q', 0).$$

But the value of the integral on the right-hand side is undefined when computed with the Bessel recursion relation [14], but is zero when computed with a Fourier table for distributions [11]:

$$\int_0^{\infty} r\, dr\, 2\pi J_0(2\pi q' r) = 0,$$

thus $\widetilde{F}_l(q', 0)$ must be obtained by a limiting procedure by approaching $\pi/2$ from above and below.

8. Some simple C_1-Radon transforms

In section 4 we have already seen the transform of a Dirac delta function concentrated at a point. The transform of an arbitrary function may be thus seen as the sum of transforms of delta functions of different strengths at points at which the function is defined. We shall give two examples.

8.1. Delta function concentrated on a straight segment

For simplicity let us assume that the linear segment of a straight line is of length L and lies parallel to the Oz axis. It is defined by the distribution:

$$l(x, y, z) = \delta(x - x_0)\delta(y)Y(L - z),$$

where $Y(x)$ is the unit step distribution of Heaviside. The C_1-Radon transform of this distribution can be evaluated using the kernel of equation (7) as

$$\widehat{C_1 l}_{(x_S, y_S, \omega)} = \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \delta(x - x_0)\delta(y)Y(L - z)\delta\left(\cos\omega\sqrt{(x - x_S)^2 + (y - y_S)^2} - z\sin\omega\right). \tag{24}$$

Successive integrations on x, y and z yield the result

$$\widehat{C_1 l}_{(x_S, y_S, \omega)} = \frac{1}{|\sin\omega|} Y\left(L - \cot\omega\sqrt{(x_0 - x_S)^2 + (y_S)^2}\right). \tag{25}$$

This is the characteristic function of the volume defined in the coordinate system $(x_S, y_S, Lt = L\tan\omega)$ by the inequality:

$$Lt > \sqrt{(x_0 - x_S)^2 + (y_S)^2}, \tag{26}$$

for given x_0. This is the interior of a circle centred at $(x_0, 0, 0)$ of radius $L\tan\omega$ in the $x_S O y_S$ plane. Alternatively if Lt is viewed as a third dimension orthogonal to $x_S O y_S$, equation (26) represents the inside volume of a circular cone with vertex at $(x_0, 0, 0)$ and opening angle $\pi/4$. Thus the C_1-Radon transform of $l(x, y, z)$ is zero everywhere except inside this cone volume where its value is $1/\sin\omega$.

8.2. Exponential function with imaginary argument

A function of the form $f(x, y, z) = \exp -2\mathrm{i}\pi(\mathbf{k}\cdot\mathbf{r})$ is essentially characterized by a vector $\mathbf{k}$ and represents a physical plane wave. We shall see that its C_1-Radon transform is also a plane wave in the $x_S O y_S$ plane with an amplitude dependent on ω and on $\mathbf{k}$. From the definition of equation (2) one has

$$\widehat{C_1 f}_{\mathbf{k},(x_S, y_S, \omega)} = \int r\sin\omega\, \mathrm{d}\psi\, \mathrm{d}r \exp(-2\mathrm{i}\pi[k_x(\rho\cos\theta + r\sin\omega\cos\psi) + k_y(\rho\sin\theta + r\sin\omega\sin\psi) + k_z r\cos\omega].) \tag{27}$$

Integration over ψ yields the Bessel function $2\pi J_0\left(2\pi r\sin\omega\sqrt{k_x^2 + k_y^2}\right)$ and the result is simply:

$$\widehat{C_1 f}(x_S, y_S, \omega) = \sin\omega \exp -2\mathrm{i}\pi(k_x x_S + k_y y_S)\mathcal{A}(\mathbf{k}, \omega), \tag{28}$$

where $\mathcal{A}(\mathbf{k}, \omega)$ is the amplitude:

$$\mathcal{A}(\mathbf{k}, \omega) = \int_0^\infty r\, \mathrm{d}r\, \mathrm{e}^{2\mathrm{i}\pi k_z r\cos\omega} 2\pi J_0\left(2\pi r\sin\omega\sqrt{k_x^2 + k_y^2}\right).$$

$\mathcal{A}(\mathbf{k}, \omega)$ is computable from a Fourier transform table [11]:

$$\begin{aligned}\mathcal{A}(\mathbf{k}, \omega) &= \left[\frac{-|k_z\cos\omega|}{\left(k_z^2\cos^2\omega - \sin^2\omega\left(k_x^2 + k_y^2\right)\right)^{3/2}}\right] \quad \text{for} \quad |k_z\cos\omega| > (k_x^2 + k_y^2)^{1/2}\sin\omega \\ &= \left[\frac{\mathrm{i}k_z\cos\omega}{\left(\sin^2\omega\left(k_x^2 + k_y^2\right) - k_z^2\cos^2\omega\right)^{3/2}}\right] \quad \text{for} \quad |k_z\cos\omega| < \left(k_x^2 + k_y^2\right)^{1/2}\sin\omega.\end{aligned} \tag{29}$$

From thereon any arbitrary function f being a Fourier superposition, its C_1-Radon transform can be obtained by integrating over 'plane waves' in the $x_S O y_S$ plane with respect to $\mathbf{k}$.

9. Relation to the standard Radon transform

We show that the C_1-cone data may be expressed in terms of standard Radon data, a result known under a restricted form in [8] for cones with two sheets. We start with the inversion formula of the three-dimensional Radon transform [1]:

$$f(\mathbf{r}) = -\frac{1}{8\pi^2}\int_{\mathbb{S}^2} \mathrm{d}\mathbf{n} \int_{-\infty}^{\infty} \mathrm{d}p\, \delta(p - \mathbf{r}\cdot\mathbf{n}) \frac{\partial^2}{\partial p^2} \widehat{Rf}_{(p\mathbf{n})}. \tag{30}$$

Now if we multiply both sides of this equation (30) by the delta function on the C_1-cone $\delta\big(\cos\omega\sqrt{(x_0-x_S)^2+(y_0-y_S)^2} - z_0\sin\omega\big)$ and integrate over $\mathrm{d}\mathbf{r}$ we would get

$$\widehat{C_1 f}_{(x_S,y_S,\omega)} = -\frac{1}{8\pi^2}\int_{\mathbb{S}^2}\mathrm{d}\mathbf{n}\int_{-\infty}^{\infty}\mathrm{d}p \frac{\partial^2}{\partial p^2}\widehat{Rf}_{(p\mathbf{n})}\int r\sin\omega\,\mathrm{d}r\,\mathrm{d}\psi$$
$$\times\,\delta(p - \overrightarrow{OS}\cdot\mathbf{n} - r(n_x\sin\omega\cos\psi + n_y\sin\omega\sin\psi + n_z\cos\omega)). \tag{31}$$

Now inserting the coordinates of $\mathbf{n}$:

$$n_x = \sin\theta\cos\phi, \qquad n_y = \sin\theta\sin\phi, \qquad n_z = \cos\theta$$

and the integration measure for $\mathbf{n}$ (i.e. $\mathrm{d}\mathbf{n} = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$), we could use the Fourier representation of the delta function of equation (5) to perform the $\mathrm{d}\psi$ integration to get the factor $2\pi J_0(2\pi q r\sin\omega\sin\theta)$.

The next integration on r is simply the integral in equation (28). After reinserting the angular values it becomes

$$\sin\omega\int_0^{\infty} r\,\mathrm{d}r\,\mathrm{e}^{2\mathrm{i}\pi q r\cos\omega\cos\theta}2\pi J_0(2\pi q r\sin\omega\sin\theta) = \frac{1}{(2\pi q)^2}\frac{1}{(\cos(\theta-\omega)\cos(\theta+\omega))^{3/2}}$$
$$\times\begin{Bmatrix} -|\cos\theta\cos\omega| \text{ if } (\cos(\theta-\omega)\cos(\theta+\omega)) > 0 \\ \mathrm{i}\cos\theta\cos\omega \text{ if } (\cos(\theta-\omega)\cos(\theta+\omega)) < 0 \end{Bmatrix}. \tag{32}$$

Next comes the integration over q. This is just a Fourier transform which yields the factor $-2\pi^2|p - (\overrightarrow{OS}\cdot\mathbf{n})|$. Putting all the partial results together, we get the final form of the C_1-Radon transform in terms of the second radial derivative of the standard Radon transform $\widehat{Rf}$:

$$\widehat{C_1 f}_{(x_S,y_S,\omega)} = \frac{\sin\omega}{16\pi^2}\iint\frac{\sin\theta\,\mathrm{d}\theta\mathrm{d}\phi}{(\cos(\theta-\omega)\cos(\theta+\omega))^{3/2}}$$
$$\times\begin{Bmatrix} -|\cos\theta\cos\omega| \text{ if } (\cos(\theta-\omega)\cos(\theta+\omega)) > 0 \\ \mathrm{i}\cos\theta\cos\omega \text{ if } (\cos(\theta-\omega)\cos(\theta+\omega)) < 0 \end{Bmatrix}$$
$$\times\int_{-\infty}^{\infty}\mathrm{d}p|p - (\overrightarrow{OS}\cdot\mathbf{n})|\frac{\partial^2}{\partial p^2}\widehat{Rf}_{(p\mathbf{n})}. \tag{33}$$

This result is further simplified if one observes that under partial integration the last integral in equation (33) can be carried out to yield the Radon transform of f:

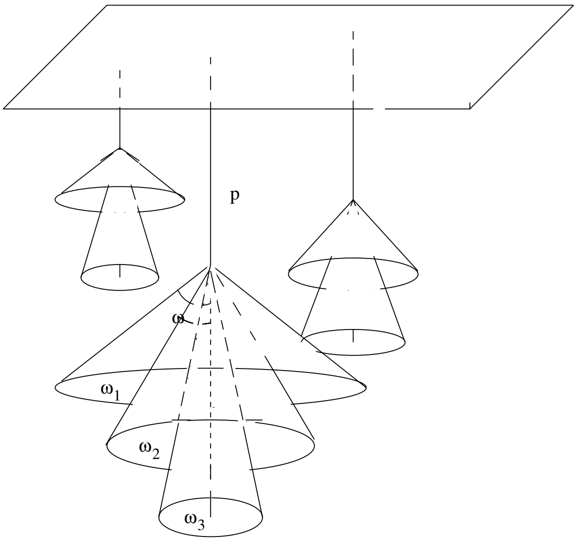

Figure 3. The compounded conical Radon transform (CCRT) with fixed axis and variable opening angle.

$$\int_{-\infty}^{\infty} \mathrm{d}p |p - (\overrightarrow{OS} \cdot \mathbf{n})| \frac{\partial^2}{\partial p^2} \widehat{Rf}_{(p\mathbf{n})} = \int_{-\infty}^{\infty} \mathrm{d}p \, \mathrm{sgn}(p - (\overrightarrow{OS} \cdot \mathbf{n})) \frac{\partial}{\partial p} \widehat{Rf}_{(p\mathbf{n})}$$
$$= \int_{-\infty}^{\infty} \mathrm{d}p \, 2\delta(p - (\overrightarrow{OS} \cdot \mathbf{n})) \widehat{Rf}_{(p\mathbf{n})} = 2\widehat{Rf}_{((\overrightarrow{OS}\cdot\mathbf{n})\mathbf{n})}. \tag{34}$$

Hence the expression of the C_1-Radon transform of f is given now in terms of Radon transforms of f on planes having normal unit vector $\mathbf{n}_{(\theta,\phi)}$ and passing through the cone vertex S.

Conversely, the inverse formula equation (23) allows us to compute the standard Radon transform from the C_1-Radon transform. We just have to integrate both sides of equation (23) with the measure $\delta(p - \mathbf{r} \cdot \mathbf{n})\,\mathrm{d}\mathbf{r}$. Hence with $\mathbf{r} = (x, y, z)$:

$$\widehat{Rf}_{(p\mathbf{n})} = \int \delta(p - \mathbf{r} \cdot \mathbf{n})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \int q\,\mathrm{d}q\,\mathrm{d}\beta \exp[2\mathrm{i}\pi q(x\cos\beta + y\sin\beta)]$$
$$\times q^2 \int_0^{\infty} t\,\mathrm{d}t\, 2\pi J_0(2\pi q z t) \int_0^{\infty} \rho\,\mathrm{d}\rho\, \widehat{G}(\rho, \beta, \epsilon(z)t). \tag{35}$$

In the integration over z, care must be exercised to account for the sign of z in the integrand. This result is analogous to the expression of the standard Radon transform of f in terms of its x-ray transform (or cone-beam transform) [9] and has helped in formulating inversion algorithms for the x-ray transform. It may also be considered as the inverse of equation (33).

10. Compounded conical Radon transform or CCRT

In this section, we introduce an extension of the C_1-Radon transform which consists in integrating the previous C_1-Radon transform along lines parallel to the Oz axis with a density function $\mu(\zeta)$. Thus the vertex of the C_1-cone is allowed to move not only in the xOy plane but also along its axis, which keeps a fixed direction (see figure 3). Such a transform occurs in emission imaging process by Compton scattered ionizing radiation [12, 13] and called the compounded conical Radon transform or CCRT.

Let $\mu(\zeta)$ be the given density function. Then the compounded conical Radon transform of a function $f(x, y, z)$ is defined by

$$\widehat{\text{CCRT}}f_\mu(x_S, y_S, \omega) = \int \mathrm{d}\zeta\, \mu(\zeta) \int r \sin\omega \,\mathrm{d}r\,\mathrm{d}\psi \\ \times f(x_S + r\sin\omega\cos\psi, y_S + r\sin\omega\sin\psi, \zeta + r\cos\omega). \tag{36}$$

Its kernel can be readily computed:

$$\mathcal{K}_\mu(x_S, y_S, \omega|x_0, y_0, z_0) = \frac{1}{\sin\omega}\mu\left(z_0 - \cot\omega\sqrt{(x_0 - x_S)^2 + (y_0 - y_S)^2}\right) \tag{37}$$

with $\rho = \sqrt{(x_0 - x_S)^2 + (y_0 - y_S)^2}$.

Assuming interchange of integrations to be valid, we define a new function h as

$$h(x_S + r\sin\omega\cos\psi, y_S + r\sin\omega\sin\psi, r\cos\omega) \\ = \int \mathrm{d}\zeta\, \mu(\zeta) f(x_S + r\sin\omega\cos\psi, y_S + r\sin\omega\sin\psi, \zeta + r\cos\omega). \tag{38}$$

In this respect we can bring the CCRT transform back to the scheme of the C_1-Radon transform acting on h instead of f. It remains to extract the true unknown function $f(x, y, z)$ from the knowledge of $h(x, y, z)$.

In fact this can be done by Fourier transform. Let $\widetilde{\mu}(\lambda)$ and $\widetilde{f}_z(x_S + r\sin\omega\cos\psi, y_S + r\sin\omega\sin\psi, \lambda)$ be the Fourier transforms of $\mu(\zeta)$ and $f(x, y, z)$, with respect to its coordinate z.

Then equation (38) may be rewritten as

$$h(x, y, z) = \int \mathrm{d}\lambda\, \widetilde{\mu}(\lambda)\widetilde{f}_z(x, y, -\lambda)\exp(-2\mathrm{i}\pi\lambda z). \tag{39}$$

Here assuming that h is determined by the inversion procedure of section 6 for all values of z, one can extract f by Fourier transforming equation (39) with respect to x, y, z to obtain:

$$\widetilde{\mu}(\lambda)\widetilde{f}(u, v, -\lambda) = \widetilde{h}(u, v, \lambda), \tag{40}$$

where $\widetilde{f}(u, v, -\lambda)$ is the three-dimensional Fourier transform of f. This is in fact a 'deconvolution' transform:

$$\widetilde{f}(u, v, w) = \frac{\widetilde{h}(u, v, -w)}{\widetilde{\mu(-w)}}.$$

Then f is recovered by inverse Fourier transform. In the work on imaging processes by scattered radiation, the function $\mu(\zeta)$ is in fact the function $1/\zeta^2$ where ζ is the distance from the cone vertex to the xOy plane, and its Fourier transform is known [11]. We must also mention that because of the assumption of spherical wave propagation the sought function f contains also a singularity $1/r^2$, where r is the distance from a running point on the cone to its vertex. The singularity can be bypassed by considering an associated CCRT transform obtained by derivation with respect to $t = \tan\omega$. The interested reader is referred to [12, 13].

11. Conclusions and perspectives

In this paper we have studied the properties of a generalized Radon transform in $\mathbb{R}^3$. Many generalizations have been proposed in the past but the cone is not a smooth surface (because of its vertex) and has not been considered in the literature of the Radon transform. However, phenomena involving Compton scattering lead naturally to integrals on circular cones since incident gamma rays scattered at a fixed scattering angle are located on a cone sheet and converge to scatterers concentrated at the cone vertex. Another generalization consists of considering other families of circular cones, for example, those with axis swinging around a point in space and having variable opening angle. The corresponding compounded conical Radon transform may also be introduced along the same lines as those in section 9 to describe a more general imaging procedure using scattered radiation. But these interesting topics will be deferred to future works as well as some important mathematical topics such as range problems, existence of an analogue to the Paley–Wiener theorem or extension to distributions.

Acknowledgment

This research is supported in part by the French Ministry of Research through grant ACI NIM TRC (2004).

References

[1] Barrett H H 1984 The Radon transform and its applications *Progress in Optics* vol 21 ed E Wolf (Groningen: North Holland) pp 219–86
[2] Cormack A M 1987 Radon's problem for some surfaces in $\mathbb{R}^n$ *Proc. AMS* **99** 305–12
[3] Cormack A M 1994 A paraboloidal Radon transform *Proc. Conf. on Seventy Five years of Radon Transforms (Vienna)* (Hong Kong: International) pp 105–9
[4] Cormack A M 1963 Representation of a function by its line integrals, with some radiological applications *J. Appl. Phys.* **34** 2722–7
Cormack A M 1964 *J. Appl. Phys.* **35** 2908–12
[5] Cree M J and Bones P J 1994 Towards direct reconstruction from a gamma camera based on Compton scattering *IEEE Trans. Med. Imaging* **13** 398–407
[6] Denecker K, Van Overloop J and Sommen F 1998 The general quadratic Radon transform *Inverse Problems* **14** 615–33
[7] Cormack A M and Quinto E T 1980 A Radon transform on spheres through the origin in $\mathbb{R}^n$ and applications to the Darboux equation *Trans. AMS* **260** 575–81
[8] Gelfand I M, Graev M I and Vilenkin N 1965 *Generalized Functions* vol V (New York: Academic)
[9] Grangeat P 1991 Mathematical framework of cone beam 3d reconstruction via the first derivative of the Radon transform *Mathematical Methods in Tomography* vol 1497 ed G T Herman, A K Louis and F Natterer (New York: Springer)
[10] Helgason S 1999 *The Radon Transform* (Berlin: Birkhäuser)
[11] Lavoine J 1963 *Transformation de Fourier* (Paris: CNRS)
[12] Nguyen M K and Truong T T 2002 On an integral transform and its inverse in nuclear imaging *Inverse Problems* **18** 265–77
[13] Nguyen M K, Truong T T, Bui H D and Delarbre J L 2004 A novel inverse problem in γ-rays emission imaging *Inverse Problems Sci. Eng.* **12** 225–46
[14] Schwartz L 1965 *Méthodes mathématiques pour les sciences physiques* (Paris: Hermann)
[15] Radon J 1917 Über die Bestimmung von Funktionnen durch ihre Integralwerte längs gewisser Mannigfaltikeiten *Ber. Verh. Sachs. Akad. Wiss. Leipzig-Math.-Natur. Kl.* **69** 262–77